The zeros of Gaussian random holomorphic functions on $\mathbb{C}^n$, and hole probability.

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Abstract

We consider a class of Gaussian random holomorphic functions, whose expected zero set is uniformly distributed over $\mathbb{C}^n$. This class is unique (up to multiplication by a non zero holomorphic function), and is closely related to a Gaussian field over a Hilbert space of holomorphic functions on the reduced Heisenberg group. For a fixed random function of this class, we show that the probability that there are no zeros in a ball of large radius, is less than $e^{-c_1 r^{2n+2}}$, and is also greater than $e^{-c_2 r^{2n+2}}$. Enroute to this result we also compute probability estimates for the event that a random function's unintegrated counting function deviates significantly from its mean.

1. **Introduction.**

Random polynomials and random holomorphic functions are studied as a way to gain insight into difficult problems such as string theory and analytic number theory. A particularly interesting case of random holomorphic functions is when the random functions can be defined so that they are invariant with respect to the natural isometries of the space in question. The class of functions that we will study are the unique Gaussian random holomorphic functions, up to multiplication by a nonzero holomorphic function, whose expected zero set is uniformly distributed on $\mathbb{C}^n$. For this class of random holomorphic functions we will determine the expected value of the unintegrated counting function for a ball of large radius and the chance that there are no zeros present. This pathological event is what is called the hole probability of a random function. In doing this we generalize a result of Sodin and Tsirelison, to n dimensions, in order to give the first nontrivial example, where the hole probability is computed in more than 1 complex variable.

The topic of random holomorphic functions is an old one which has many results from the first half of the twentieth century, and is recently experiencing a second renaissance. In particular Kac determined a formula for the expected distribution of zeros of real polynomials in a certain case, [6]. This work was generalized throughout the years, and a terse geometric proof, and some consequences are presented by Edelman and Kostlan, [3]. An excellent reference for other results regarding the general properties of

random functions is Kahane's text, [7]. One series of papers, by Offord, is particularly relevant to questions involving the hole probability of random holomorphic functions and the distribution of values of random holomorphic functions, [10], [11], although neither is specifically used in this paper. Recently, there has been a flurry of interest in the zero sets of random polynomials and holomorphic functions which are much more natural objects than they may initially appear. For example Bleher, Shiffman and Zelditch show that for any positive line bundle over a compact complex manifold, the random holomorphic sections to L^N (defined intrinsically) have universal high N correlation functions, [1].

In addition to a plethora of results describing the typical behavior, there have also been several results in 1 (real or complex) dimension for Gaussian random holomorphic functions where the hole probability has been determined. For a specific class of real Gaussian polynomials of even degree 2n, Dembo, Poonen, Shao and Zeitouni have shown that for the event where there are no real zeros, E_n, the $\lim_{n\to\infty} \frac{Prob(E_n)}{\log(n)} n^{-b} = -b,\ b \in [0.4, 2]$, [2]. Hole probability for the complex zeros of a Gaussian random holomorphic function is a quite different problem. Let $Hole_r = \{f$, in a class of holomorphic functions, such that $\forall z \in B(0,r),\ f(z) \neq 0\}$. For the complex zeros in one complex dimension, there is a general upper bound for the hole probability: $Prob(Hole_r) \leq e^{-c\mu(B(0,r))},\ \mu(z) = E[Z_{\psi_\omega}]$ as in theorem 3.1, [14]. In one case this estimate was shown by Peres and Virag to be sharp: $Prob(Hole_r) = e^{-\frac{\mu(B(0,r))}{24} + o(\mu(B(0,r)))}$, [12]. These last two results on hole probability might suggest that when the random holomorphic functions are invariant with respect to the local isometries, thus ensuring that $E[Z_\omega]$ is uniformly distributed on the manifold, the rate of decay of the hole probability would be the same as that which would be arrived at if the zeros where distributed according to a Poisson process. However, as the zeros repel in 1 dimension, [4], one might expect there to be a quicker decay for hole probability of a random holomorphic function. This is the case for random holomorphic functions whose expected zero set is uniformly distributed on $\mathbb{C}^1$, [15] :

$$Prob(Hole_r) \leq e^{-c_1 r^4} = e^{-c\mu(B(0,r))^2},\ and\ Prob(Hole_r) \geq e^{-c_2 r^4} = e^{-c\mu(B(0,r))^2}$$

The random holomorphic functions that we will study, can be written as

$$\psi_\omega(z_1,\ z_2, \ldots,\ z_n) = \sum_{j \in \mathbb{N}^n} \omega_j \frac{z_1^{j_1} \cdot z_2^{j_2} \cdot \ldots \cdot z_n^{j_n}}{\sqrt{j_1 \cdot j_2 \cdot \ldots \cdot j_n}} = \sum_{j \in \mathbb{N}^n} \omega_j \frac{z^j}{\sqrt{j!}}$$

where ω_j are independent identically distributed standard complex gaussian random variables, and a.s. are holomorphic on $\mathbb{C}^n$. The second form is just the standard multi-index notation, and will frequently be used from here on out. Random holomorphic functions of this form are a natural link between Hilbert spaces of holomorphic functions on the reduced Heisenberg group and a similar Gaussian Hilbert Space. Further, these random functions will be the unique class (up to multiplication by a nonzero entire function) whose expected distribution of the zero set is:

$$E[Z_\omega] = \frac{i}{2\pi}(dz_1 \wedge d\overline{z}_1 + dz_2 \wedge d\overline{z}_2 + \ldots + dz_n \wedge d\overline{z}_n).$$

The two main results of this paper are:

Theorem 1.1. *If*

$$\psi_\omega(z_1, z_2 \ldots, z_n) = \sum_j \omega_j \frac{z_1^{j_1} z_2^{j_2} \ldots z_n^{j_n}}{\sqrt{j_1! \cdot j_n!}},$$

where ω_j are independent identically distributed complex Gaussian random variables,
then for all $\delta > 0$, there exists $c_{3,\delta} > 0$ and $R_{n,\delta}$ such that for all $r > R_{n,\delta}$

$$Prob\left(\left\{\omega : \left|n_{\psi_\omega}(r) - \frac{1}{2}r^2\right| \geq \delta r^2\right\}\right) \leq e^{-c_{3,\delta} r^{2n+2}}$$

where $n_{\psi_\omega}(r)$ is the unintegrated counting function for ψ_ω.

Theorem 1.2. *If*

$$Hole_r = \{\omega : \forall z \in B(0,r),\ \psi_\omega(z) \neq 0\},$$

then there exists $R_n,\ c_1,$ and $c_2 > 0$ such that for all $r > R_n$

$$e^{-c_2 r^{2n+2}} \leq Prob(Hole_r) \leq e^{-c_1 r^{2n+2}}$$

The proof of Theorems 1.1 and 1.2, will use techniques from probability theory, several complex variables and an invariance rule for Gaussian random holomorphic functions which is derived from isometries of the reduced Heisenberg group. These results, using the mainly same techniques, were already proven in the case where n=1 by Sodin and Tsirelison, [15].

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2. **The link between random holomorphic functions, Gaussian Hilbert spaces and the reduced Heisenberg group.**

To develop the notion of a random holomorphic function on $\mathbb{C}^n$ we will need a way to place a probability measure on a space of holomorphic functions on $\mathbb{C}^n$. The definition we will use is that a random holomorphic function is a representative of a Gaussian field between two Hilbert spaces on the reduced Heisenberg group. Through this definition we will prove the crucial Lemma 4.3 which gives a nice law to determine how random holomorphic functions behave under translation. Additionally, this definition is equivalent to defining a random holomorphic function as $\psi_\omega(z) = \sum_{j\in\mathbb{N}^n} \omega_j \left(\frac{z_1^{j_1} z_2^{j_2} \ldots z_n^{j_n}}{\sqrt{j_1! j_2! \ldots j_n!}}\right)$, where ω_j are independent identically distributed standard complex Gaussian random variables.

We will start with the concept of a Gaussian Hilbert Space, as presented by Janson, [5],

Definition 2.1. *A Gaussian Linear space, G, is a linear space of random variables, defined on a probability space $(\Omega, d\nu)$, such that each variable in the space is Gaussian random variable.*

Definition 2.2. *A Gaussian Hilbert Space, G is a Gaussian linear space that is complete with respect to the $L^2(\Omega, d\nu)$ norm.*

We will shortly apply these definitions to a Hilbert space of CR-holomorphic functions on the reduced Heisenberg group. The Heisenberg Group, as a manifold is nothing other than $\mathbb{C}^n \times \mathbb{R}$, and the reduced Heisenberg group is the circle bundle: $X = H^n_{red} = \left\{(z,\alpha), z \in \mathbb{C}^n, \alpha \in \mathbb{C},\ |\alpha| = e^{\frac{-|z|^2}{2}}\right\}$.

Consider holomorphic functions of the Heisenberg group, which are linear with respect to the $n+1^{st}$ variable. The restriction of these functions define functions on X. For functions on X there is the following inner product:

$$\begin{aligned}(F,G) = \int_X F\overline{G} &= \frac{1}{\pi^n}\int_X f(z)\overline{g(z)}|\alpha|^2 d\theta(\alpha)dm(z)\\ &= \frac{1}{\pi^n}\int_{\mathbb{C}^n} f(z)\overline{g(z)}e^{-|z|^2}dm(z)\end{aligned}$$

Here dm is Lebesque measure. With respect to this inner product,

$H_X = \{F \in \mathcal{O}(\mathbb{C}^{n+1}),\ F(z,\alpha) = f(z)\alpha,\ f \in \mathcal{O}(\mathbb{C}^n)\}$ is a Hilbert Space, and $H_X \cong H^2(\mathbb{C}^n,\ e^{-|z|^2}dm)$, as Hilbert Spaces.

Proposition 2.3. *For H_X, $\left\{\frac{z_1^{j_1}\cdot z_2^{j_2}\cdot\ldots z_n^{j_n}}{\sqrt{j_1!\cdot\ldots\cdot j_n!}}\alpha\right\}_{j\in\mathbb{N}^n} = \{\psi_j(z)\alpha\}_{\mathbb{N}^n}$ is an orthonormal basis.*

The proof of this proposition is a straight forward computation.

The isometries of the reduced Heisenberg group will play a crucial role in my computation of the hole probability. These isometries are of the form:

$$\tau_{(\nu,\alpha)} : H^m_{red} \to H^m_{red}$$

$$\tau_{(\nu,\alpha)} : (\zeta,\beta) \to (\zeta+\nu, e^{-\zeta\overline{\nu}}\alpha\beta)$$

The inner product on X is invariant with respect to the Heisenberg group law:

$$\begin{aligned}\tau^*\langle F,G\rangle &= \frac{1}{\pi^n}\int\int f(\zeta+\nu)\overline{g(\zeta+\nu)}|\beta\alpha e^{-\zeta\overline{\nu}}|^2 d\theta\ dm(\zeta)\\ &= \frac{1}{\pi^n}\int f(\zeta+\nu)\overline{g(\zeta+\nu)}e^{-|(\zeta+\nu)|^2}dm(\zeta+\nu)\\ &= \frac{1}{\pi^n}\int f(\zeta)\overline{g(\zeta)}e^{-|\zeta|^2}dm(\zeta)\end{aligned}$$

As such for $\alpha = e^{-\frac{|z|^2}{2}}$,

$$\tau^*(\alpha\psi_j(z)) = (\alpha e^{-\frac{1}{2}|\zeta|^2 - z\overline{\zeta}}\psi_j(z+\zeta)) = e^{-\frac{1}{2}|z+\zeta|^2 - i\cdot Im(z\overline{\zeta})}\psi_j(z+\zeta)$$

and the collection of these, $\{e^{-\frac{1}{2}|z+\zeta|^2-i\cdot Im(z\overline{\zeta})}\psi_j(z+\zeta)\}$, is another orthonormal basis for H_X, as the inner product is invariant with respect to the group law.

Example 2.4. *(Gaussian Hilbert Spaces)*

Let $G'_{H_X} = Closure(Span\,(\{\omega_j\psi_j(z)\alpha\}_{\mathbb{N}^n}))$, where the closure is taken with respect to the norm $E[(\|\cdot\|_{H_X})^2]^{\frac{1}{2}}$ and where ω_j are independent identically distributed standard complex Gaussian random variables. G'_{H_X} is not a Gaussian Hilbert space but is isometric to $G_{H_X} = Closure(Span\,(\{\omega_j\}_{\mathbb{N}^n}))$, which is.

Of course,

$$\begin{array}{ccccc} H_X & \to & G'_{H_X} & \to & G_{H_X}\\ \sum a_j\psi_j(z)\alpha & \longmapsto & \sum a_j\omega_j\psi_j(z)\alpha & \longmapsto & \sum a_j\omega_j\end{array}$$

are isometries.

G'_{H_X} is in many ways more natural then G_{H_X}, and is closely related to random holomorphic functions.

Definition 2.5. *A Gaussian field is a linear isometry $L : H_X \to G_{H_X}$. As such, for all $f\in H_X, L[f] = X_f$, a standard complex Gaussian random variable with $Var = \|f\|^2_{H_X}$.*

Definition 2.6. *A Gaussian random function, is a representative for a Gaussian field L. In other words,*

$$if\ \ f \in H_X, L[f] = \langle \varphi_\omega, f \rangle_{H_X}$$

Remark 2.7. *(Random holomorphic functions on $\mathbb{C}^n$ or (equivalently) Gaussian fields and functions between H_X and G_{H_X})*

Let $\psi_\omega(z) = \sum_{j \in \mathbb{N}} \omega_j \psi_j(z) = \sum_{j \in \mathbb{N}} \omega_j \frac{z^j}{\sqrt{j!}}$ *, ω_j independent identically distributed standard complex Gaussian random variables. This will shortly be shown to a.s. be a holomorphic function on $\mathbb{C}^n$ by Theorem 2.8.*

Note that for $\alpha f(z) \in H_X,\ f(z) = \sum a_j \psi_j(z),\ \{a_j\} \in \ell^2$, $\langle \alpha\psi_\omega(z), \alpha f(z) \rangle = \sum \overline{a}_j \omega_j$, which is a complex Gaussian random variable with variance $\sum |a_j|^2 = \|f\|_{H_{\mathbb{C}^n}}$, hence $\alpha\psi_\omega(z)$ is a random CR-holomorphic function on X.

The variable α will be useful when we change bases. This occurs when we look at how random functions behave with respect to translation (Lemma 4.3). Abusing notation we will frequently drop it and we will call $\psi_\omega(z)$ a random holomorphic function on $\mathbb{C}^n$.

There is a simple condition for when a function of the form $\sum \omega_j \psi_j(z)$ is a holomorphic condition, where ω_j are independent identically distributed complex Gaussian Random variables.

Theorem 2.8. *Let $\{\omega_j\}_{j \in \mathbb{N}}$ be a sequence of independent identically distributed, standard complex Gaussian random variables. If for $j \in \mathbb{N}$, $\psi_j(z) \in \mathcal{O}(\Omega)$, and for all compact $K \subset \Omega$, $\sum_{j \in \mathbb{N}} \max_{z \in K} |\psi_j(z)|^2 < \infty$*

then for a.a.-ω, $\sum_{j \in \mathbb{N}} \omega_j \psi_j(z)$ defines a holomorphic function on Ω.

This theorem can be proved easily by adapting a similar proof of convergence of "random sums" from [7].

Theorem 2.9. *If L is a Gaussian field, $L : H_X \to G_{H_X}$, and $\{\phi_j\}$ is an orthonormal basis for H_X*

then L can be written as: $L[\cdot] = \langle \phi_X, \cdot \rangle$, where $\phi_X(z) = \sum X_i \phi_i$, and $\{X_i\}$ is a set of independent identically distributed standard Gaussian random variables.

Proof. Let L be a Gaussian random functional.
Let $X_1 = L[\phi_1],\ X_2 = L[\phi_2],\ \dots,\ X_j = L[\phi_j], \dots$

We must only show that X_j are independent identically distributed Gaussian Random variables, hence it suffices to prove independence as by the definition of Gaussian random field, $X_i,\ X_j$ are jointly normal, as $L\left[\sum a_j\psi_j\right] = \sum a_j X_j$ is normal.
For $i \neq j$:

$$\begin{aligned} 2 &= E[|X_i + X_j|^2] = E[|X_i|^2] + E[|X_j|^2] + E[X_i\overline{X}_j] + E[X_j\overline{X}_i] \\ &= 2 + E[X_i\overline{X}_j] + E[X_j\overline{X}_i] \end{aligned}$$

Hence $Re(E[X_i\overline{X}_j]) = 0 = Im(E[X_i\overline{X}_j])$,

The result then follows.

□

3. **Common Results.**

Let us briefly review properties of the zeros of random holomorphic functions. An elementary way to view the zeros of a holomorphic function is as a set: $Z_f = f^{-1}(\{0\})$, but this will be insufficient for my purpose, and we will instead view it as a (1,1) current. For M^n an n dimensional manifold, and $f \in \mathcal{O}(M),\ f : M^n \to \mathbb{C},\ f^{-1}(\{0\})$ is a divisor. Hence the regular points of Z_f are a manifold, and by taking restriction we identify forms in $D_M^{(n-1,n-1)}$ with ones in $D_{Z_{f,reg}}^{(n-1,n-1)}$. As $Z_{f,reg}$ is an n-1 complex manifold, $\displaystyle\int_{Z_{f,reg}}$ is a (1,1) current on M, which we will denote Z_f (abusing notation). As the singularities occur in real codimension 2. $Z_f = Z_{f,reg}$, and in general: if $f \in \mathcal{O}(M^n)$, M an n complex manifold, then $Z_f = \frac{i}{2\pi}\partial\overline{\partial}\log|f|^2$, as $(1,1)$ currents on M.

Before we classify the atypical hole probability, we shall first describe the expected behavior. Many various forms of the following theorem have been proven, [3], [6] and [14]. For my purposes it is important that the proof is valid in n-dimensions, and for infinite sums. Many of the proofs resemble this one. After a conversation with Steve Zelditch, I was able to simplify a previously complicated argument into the current form. This simplification is already known to other researchers including Mikhail Sodin.

For the following theorem let $\psi_j : \Omega \to \mathbb{C},\ j \in \Lambda,\ \Lambda = \{0, 1, 2, \dots, n\}$ or $\Lambda = \mathbb{N}$, be a sequence of holomorphic functions on a domain of an n manifold to $\mathbb{C}$.

Theorem 3.1. *If* $E[|\psi_\omega|^2] = \sum |\psi_j(z)|^2$ *converges locally uniformly in* Ω *then* $E[Z_\omega] = \frac{i}{2\pi}\partial\overline{\partial}\log||\psi(z)||^2_{\ell^2}$

Proof. Let $\beta \in D^{n-1,n-1}(\Omega)$

To simplify the notation, let $\beta = \phi\ dz_2 \wedge d\overline{z}_2 \wedge \ldots \wedge dz_n \wedge d\overline{z}_n$.

$$\begin{aligned} \langle Z_{\psi_\omega}, \beta\rangle &= \langle \tfrac{i}{2\pi}\partial\overline{\partial}\log(|\psi_\omega(z)|^2), \beta\rangle \\ &= \langle \tfrac{i}{2\pi}\log\left(|\psi_\omega(z)|^2\right), \partial\overline{\partial}\beta\rangle \\ &= \langle \tfrac{i}{2\pi}\left(\log(||\psi(z)||^2_{\ell^2}) + \log\left(\tfrac{|\psi_\omega(z)|^2}{||\psi(z)||_{\ell^2}}\right)\right), \partial\overline{\partial}\beta\rangle \end{aligned}$$

Taking the expectation of both sides we compute:

$$\begin{aligned} E[\langle Z_{\psi_\omega}, \beta\rangle] = \tfrac{1}{2\pi} \quad & \int_\omega \int_{z\in\Omega} \log(||\psi(z)||^2_{\ell^2}) \tfrac{\partial^2\phi}{\partial z_1 \partial\overline{z}_1}\ dm(z)\ d\nu(\omega) \\ & + \tfrac{1}{2\pi}\int_\omega \int_{z\in\Omega} \log\left(\tfrac{|\psi_\omega(z)|^2}{||\psi(z)||^2_{\ell^2}}\right) \tfrac{\partial^2\phi}{\partial z_1 \partial\overline{z}_1}\ dm(z)\ d\nu(\omega) \end{aligned}$$

The first term is the desired result (which by assumption is integrable and finite), while the second term will turn out to be zero. We first must establish that it is in fact integrable:

$$\begin{aligned} \int_{z\in\Omega}\int_\omega |\log\left(\tfrac{|\psi_\omega(z)|^2}{||\psi(z)||^2_{\ell^2}}\right) \quad & \tfrac{\partial^2\phi}{\partial z_1\partial\overline{z}_1} d\nu(\omega)|\ dm(z) \\ & \le c\int_{z\in K}\int_\omega \left|\log\left(\tfrac{|\psi_\omega(z)|^2}{||\psi(z)||^2_{\ell^2}}\right)\ d\nu(\omega)\right|\ dm(z) \\ & = c\int_{z\in K}\int_\omega |\log(|\omega'|^2)|\ d\nu(\omega')\ dm(z) \end{aligned}$$

where ω' is a standard centered Gaussian ($\forall z$), thusly proving integrability as:

$$\int_{z\in\Omega}\int_\omega \left|\log(\frac{|\psi_\omega(z)|^2}{||\psi(z)||^2_{\ell^2}})\right| \le C\int_{|x|<1}|\log(x)|dm(x) + c\int_{|x|>1}|xe^{-x^2}|dm(x) \le c$$

Finally,

$$\int_\Omega \beta\wedge E[Z_{\psi_\omega}] = \frac{i}{2\pi}\int_\Omega \partial\overline{\partial}\beta\log(\|\psi(z)\|^2_{\ell^2}) \ + \ \int_\Omega C\frac{\partial^2\phi}{\partial z_i\partial\overline{z}_j}dm(z)$$

$$= \frac{i}{2\pi}\int_\Omega \partial\overline{\partial}\beta\wedge\log(||\psi(z)||^2_{\ell^2})$$

□

Corollary 3.2. *For* ψ_ω *a random holomorphic function on* $\mathbb{C}^n$,

$$E[Z_{\psi_\omega}] = \frac{i}{2\pi}(dz_1\wedge d\overline{z}_1 + dz_2\wedge d\overline{z}_2 + \ldots + dz_n\wedge d\overline{z}_n)$$

In Theorem 3.1, we proved that the expected zero set is determined by the variance of a of the random function when evaluated at a point. More can be said:

Theorem 3.3. *For gaussian analytic functions the expected zero set determines the process uniquely (up to multiplication by nonzero holomorphic functions) on a simply connected domain.*

This theorem is proven in one dimension by Sodin [14], and the same proof works in n-dimensions.

4. **Invariance of Gaussian random functions with respect to the isometries of the reduced Heisenberg group.**

The invariance property of the random function in question with respect to the reduced Heisenberg group's isometries plays a central role in proving that:

$$\max_{z\in\partial B(0,r)}\left(\log(|\psi_\omega(z)|)-\frac{1}{2}|z|^2\right)=\max_{z\in\partial B(\zeta,r)}\left(\log(|\psi_\omega(z)|)-\frac{1}{2}|z|^2\right)$$

This invariance property, which was known and used in the 1 dimensional case, and makes sense (from the view that $\forall(z,\alpha)\in X,\ \alpha\psi_\omega(z))$ defines a standard complex Gaussian random variable for any fixed z. Apparently, however, there was no proof in the literature until recently, [9]. I also independently came up with this same result by using the properties of the Heisenberg group, and this is presented here.

This next result is that a random holomorphic function is well defined independent of basis chosen, and also will be shortly restated in order to give an important translation law for random holomorphic functions of $\mathbb{C}^n$.

Lemma 4.1. *If $\{\phi_j\}_{j\in\Lambda}$ is an orthonormal basis for H_X*

then there exists $\{\omega_j'\}_{j\in\Lambda}$ independent identically distributed standard complex Gaussian random variables such that for all $(z,\alpha)\in H^n_{red}$,

$$\alpha\psi_\omega(z)=\phi_{\omega'}(z,\alpha),\ \ a.s.$$

where $\phi_{\omega'}(z,\alpha)=\sum\omega_j'\phi_j(z,\alpha)$.

Proof. For all $j\in\mathbb{N}$, let $\omega_j'=\langle\alpha\psi_\omega(z),\phi_j(z,\alpha)\rangle_{H_X}$, which is a standard complex Gaussian random variable by Theorem 2.9. Further, for $j\neq k,\ \omega_j'$ and ω_k' are independent.

Let $f\in H_X\Rightarrow f=\sum a_j\phi_j,\ \{a_j\}_{j\in\mathbb{N}}\in\ell^2$
We now demonstrate that $\phi_{\omega'}=\psi_\omega$ as a Gaussian field:

$$\begin{aligned} \langle \phi_{\omega'}, f \rangle &= \sum \overline{a}_j \omega'_j \\ \langle \psi_\omega, f \rangle &= \langle \psi_\omega, \sum a_j \phi_j \rangle \\ &= \sum \overline{a}_j \langle \psi_\omega, \phi_j \rangle \\ &= \sum \overline{a}_j \omega'_j \end{aligned}$$

As $H_X = H_X^*$, for evaluation maps $eva_{(z_0,\alpha_0)} = \langle \sum b_n \phi_j, \cdot \rangle$, $\sum b_n \phi_j \in H_X$ and therefore by the above work:

$$\langle \phi_{\omega'}, \sum \overline{b}_j \phi_j \rangle = \sum \omega'_j b_j = \sum \omega'_j \phi_j(z_0, \alpha_0) = \sum \omega_j \alpha_0 \psi_j(z_0) \qquad \square$$

Definition 4.2. *A Gaussian random function is invariant with respect to τ if both $\langle \cdot, \psi_\omega \rangle$ and $\langle \cdot, \tau^* \psi_\omega \rangle$ induce the same Gaussian field.*

A random CR-holomorphic function on X, will be invariant with respect to isometries of X. These will in turn be important for random holomorphic functions on $\mathbb{C}^n$, which is illustrated in the following simple but important lemma.

Let $\tau(z, \alpha) = (z + \zeta, e^{-z\overline{\zeta}} \alpha\beta),\ |\beta| = e^{-\frac{|\zeta|^2}{2}}$. Recall that τ is an isometry of H_X.

Lemma 4.3. *For all $z \in \mathbb{C}^n$, there exists ω'_j independent identically distributed standard complex Gaussian random variables, such that*

$$e^{-\frac{1}{2}|z|^2} \psi_\omega(z) = e^{-\frac{1}{2}|z+\zeta|^2 - i \cdot Im(z\overline{\zeta})} \psi_{\omega'}(z + \zeta)$$

Proof. Here, ζ is any fixed complex number and we set $\beta = e^{-|\zeta|^2}$. Both $\{\alpha\psi_j(z)\}$ and $\{\tau^*(\alpha\psi_j(z))\}$ are orthonormal bases of H_X, and they therefore induce the same Gaussian random function, as these are well defined independent of basis by Lemma 4.1. Hence

$$\begin{aligned} \alpha\psi_\omega(z) &= \alpha\beta e^{-z\overline{\zeta}} \psi_{\omega'}(z + \zeta) \\ e^{-|z|^2} \psi_\omega(z) &= e^{-\frac{1}{2}(|z|^2 + |\zeta|^2 + 2z\overline{\zeta})} \psi_{\omega'}(z + \zeta) \\ &= e^{-\frac{1}{2}|z+\zeta|^2 - i \cdot Im(z\overline{\zeta})} \psi_{\omega'}(z + \zeta) \end{aligned}$$

$\square$

Corollary 4.4. *The random variable:* $\max_{z \in \partial B(0,r)} \left(\log(|\psi_\omega(z)|) - \frac{1}{2}|z|^2 \right)$ *is invariant with respect to τ^*. In other words*

$$\max_{z \in \partial B(0,r)} \left(\log(|\psi_\omega(z)|) - \frac{1}{2}|z|^2 \right) = \max_{z \in \partial B(\zeta,r)} \left(\log(|\psi_\omega(z)|) - \frac{1}{2}|z|^2 \right)$$

Proof. This corollary just specializes the previous lemma as,

$$\begin{aligned}
\max_{z\in\partial B(0,r)}\left(\log|\psi_\omega(z)|-\frac{1}{2}|z|^2\right) &= \max_{z\in\partial B(0,r)}\left(\log|\psi_\omega(z)|-\log(e^{\frac{1}{2}|z|^2})\right)\\
&= \max_{z\in\partial B(0,r)}\log(|\alpha\psi_\omega(z)|),\ |\alpha|=e^{-\frac{1}{2}|z|^2}\\
&= \max_{z\in\partial B(0,r)}\log(\tau^*(|\alpha\psi_\omega(z)|))\\
&= \max_{z\in\partial B(0,r)}\log(|\beta\psi_\omega(z+\zeta)|),\ |\beta|=e^{-\frac{1}{2}|z+\zeta|^2}\\
&= \max_{z\in\partial B(\zeta,r)}\log((|\beta'\psi_\omega(z)|)),\ |\beta'|=e^{-\frac{1}{2}|z|^2}\\
&= \max_{z\in\partial B(\zeta,r)}\left(\log|\psi_\omega(z)|-\frac{1}{2}|z|^2|\right)
\end{aligned}$$

□

5. An estimate for the growth rate of random holomorphic functions on the reduced Heisenberg group.

In this section we begin working towards my main results. Lemma 5.4 is interesting in and of itself as it proves that random functions for H_X are of finite order 2, a.s. From hence forth we will work with $\mathbb{C}^n$, for any one fixed n.

Definition 5.1. *A family of events $\{E_r\}_{r\in\mathbb{R}^+}$, dependent on r, will be called a small family of events if exists R, and $c>0$, such that for all $r>R$, $Prob(E_r)\le e^{-cr^{2n+2}}$.*

We will be using properties of Gaussian random holomorphic functions to deduce typical properties of functions, and the size of the family of events where these typical properties will not work will always be small.

Let $M_{r,\omega}=\max_{\partial B(0,r)}\log|\psi_\omega(z)|$

We will be able to compute this, adapting a strategy that Sodin and Tsirelison, [15], used to solve the analogous 1 dimension problem, by using the Cauchy Integral Formula in conjunction with some elementary probability theory and computations:

Lemma 5.2. *Let ω be a standard complex Gaussian RV (mean 0, Variance 1), with a probability distribution function $d\nu(\omega)$*
then: *a-i)* $\nu(\{\omega:|\omega|\ge\lambda\})=e^{-\lambda^2}$
a-ii) $\nu(\{\omega:|\omega|\le\lambda\})=1-e^{-\lambda^2}\in[\frac{\lambda^2}{2},\lambda^2], if \lambda\le 1$

b) If $\{\omega_j\}_{j\in\mathbb{N}^n}$ is a set of of independent identically distributed standard Gaussian random variables, then $\nu(\{\omega:|\omega_j|<(1+\varepsilon)^{|j|}\})=c>0$.

Here, and throughout this paper for $j \in \mathbb{N}^n,\ |j| := \sum j_i$

Lemma 5.2-a) is a straight forward computation using that the probability distribution for a standard complex Gaussian.

Proof. of b) $\nu(\{\omega_j : |\omega_j| < (1+\varepsilon)^{|j|}\}) = 1 - e^{-(1+\varepsilon)^{2|j|}}$

$$\big(\nu(\{\omega : |\omega_j| < (1+\varepsilon)^{|j|}\}) > 0\big) \ \Leftrightarrow \prod_{j\in\mathbb{N}^n, |j|=0,}^{|j|=\infty} 1 - e^{-(1+\varepsilon)^{2|j|}} = c$$

$$\Leftrightarrow c \sum |j|^n |\log\left(1 - e^{-(1+\varepsilon)^{2|j|}}\right)| < \infty$$

as there are about $c|j|^n,\ j \in \mathbb{N}^n$ with a fixed value of $|j|$.

$$\forall |x| < 1,\ \log(1-x) = -\int \sum (x)^m = \sum_{m\geq 0} \frac{x^{m+1}}{m+1}$$

Therefore, $\sum |j| \left|\log\left(1 - e^{-(1+\varepsilon)^{2|j|}}\right)\right| \leq c \sum_m m^n e^{-(1+\varepsilon)^{2m}} < \infty$ □

The following lemma is needed twice in this paper, including in the proof of Lemma 5.4.

Lemma 5.3. *If* $j \in \mathbb{N}^{+,n}$ *then* $\frac{|j|^{|j|}}{j^j} \leq n^{|j|}$

Proof. Let $u_k = \frac{j_k}{|j|} \geq 0$, hence $\sum_{k=1}^n u_k = 1$.
As $\sum u_k \delta_{u_k}$ is a probability measure:

$$\begin{aligned}\sum_{j=1}^n u_k \log\left(\frac{1}{u_k}\right) &\leq \log \sum \frac{1}{u_k} u_k, \text{ by Jensen's inequality.}\\ &= \log(n)\end{aligned}$$

$$\begin{aligned}\text{Hence, } n^{|j|} &\geq \prod_k (u_k)^{-|j|u_k}\\ &= \tfrac{|j|^{|j|}}{j^j}\end{aligned}$$

□

Lemma 5.4. *(Probabilistic Estimate on the Rate of growth of the maximum of a random function on* $\mathbb{C}^n$*)*
For all $\delta > 0$*,*

$$E_{r,\delta} := \left\{\omega : \left|\frac{log(M_{r,\omega})}{r^2} - \frac{1}{2}\right| \geq \delta\right\} \ is\ a\ small\ family\ of\ events$$

Proof. We will first prove that: $\nu(\{\omega : \frac{log(M_{r,\omega})}{r^2} \geq \frac{1}{2} + \delta\} \leq e^{-c_{\delta,1} r^{2n+2}}$ and we will prove this by specifying a set of measure almost 1 where the max

grows at the appropriate rate.

Let Ω_r be the event where: $i)\ |\omega_j| \le e^{\frac{\delta r^2}{4}},\quad |j| \le 2e \cdot n \cdot r^2$
$ii) |\omega_j| \le 2^{\frac{|j|}{2}},\quad |j| > 2e \cdot n \cdot r^2$

$$\begin{aligned}
\nu(\Omega_r^c) &\le \sum_{|j| \le 2e \cdot nr^2} \nu(\{|\omega_j| > e^{\frac{\delta r^2}{4}}\}) + \sum_{|j| > 2e \cdot nr^2} \nu(\{|\omega_j| > 2^{\frac{|j|}{2}}\}) \\
&\le c_n r^{2n} e^{\left(-e^{\frac{\delta r^2}{2}}\right)} + \sum_{|j| > 2e \cdot nr^2} e^{-2^{|j|}} \\
&\le e^{-e^{cr^2}} + ce^{-2^{cr^2}},\ \forall r > R_0 \\
&\le e^{-e^{cr}}
\end{aligned}$$

We now have that Ω_r^c is contained in a small family of events (and in fact could make a stronger statement on the rate of decay in terms of r). It now remains for me to show that $\forall \omega \in \Omega_r,\ \frac{\log|M_{r,\omega}|}{r^2} \le \frac{1}{2} + \frac{1}{2}\delta$.

$\forall \omega \in \Omega_r$, we have that:

$$M_{r,\omega} \le \sum_{|j|=0}^{|j| \le 4e \cdot n(\frac{1}{2}r^2)} |\omega_j| \frac{|z|^j}{\sqrt{j!}} + \sum_{|j| > 4e \cdot n(\frac{1}{2}r^2)} |\omega_j| \frac{|z|^j}{\sqrt{j!}} = \sum^1 + \sum^2$$

Using the Cauchy-Schwartz inequality:

$$\begin{aligned}
\sum^1 &\le (e^{\frac{1}{4}\delta r^2}) \sqrt{c(r^2)^n} \left(\sum_j \frac{|z^{2j}|}{j!} \right)^{\frac{1}{2}} \\
&\le c_n e^{\frac{\delta r^2}{4}} r^n e^{\frac{1}{2}r^2} \\
&\le e^{(r^2)(\frac{1}{2} + \frac{1}{3}\delta)},\ \forall r > R_{n,\delta}.
\end{aligned}$$

$$\begin{aligned}
\sum^2 &\le \sum_{|j| > 4e \cdot nr^2} (2)^{\frac{|j|}{2}} \frac{|z^j|}{\sqrt{j!}} \\
&\le \sum_{|j| > 4e \cdot nr^2} (2)^{\frac{|j|}{2}} \left(\frac{|j|}{4en} \right)^{\frac{|j|}{2}} \prod_k \left(\frac{e}{j_k} \right)^{\frac{j_k}{2}}, \text{ by Sterling's Formula} \\
&\le C, \text{ by Lemma 5.3.}
\end{aligned}$$

Hence, $\forall \omega \in \Omega_r,\ \log(M_{r,\omega}) \le (\frac{1}{2} + \frac{1}{2}\delta) r^2$

It now remains for me to show that:

$$\forall \delta < \Delta,\ \nu \left(\left\{ \omega : \frac{log(M_{r,\omega})}{r^2} \le \frac{1}{2} - \delta \right\} \right) \le e^{-c_{\delta,2} r^{2n+2}}$$

which we will do by using Cauchy's integral formula to transfer information on $M_{r,\omega}$ to individual coefficients ω_j. It suffices to prove this result only for small δ as $\delta < \delta' \Rightarrow E_{\delta',r} \subset E_{\delta,r}$. The constant Δ can be explicitly determined.

It will be most convenient to prove this result for the polydisk, where the Cauchy Integral Formula applies. The notation for the polydisk is the standard one: $P(0,r) := \{z \in \mathbb{C}^n : \ \forall i, \ |z_i| < r\}$

Let $M'_{r,\omega} = \max_{z \in P(0,r)} |\psi_\omega(z)|$

The corresponding claim for a poly disk is that:

$$M'_{r,\omega} \geq \frac{n}{2} r^2 - \delta r^2$$

except for a small family of events.

We will now look at the probability of the event consisting of ω such that:

$$\log(M'_{r,\omega}) \leq \left(\frac{n}{2} - \delta\right) r^2$$

By Cauchy's Integral Formula: $\left|\frac{\partial^j \psi_\omega}{\partial z^j}\right|(0) \leq j! M'_{r,\omega} r^{-|j|}$

By direct computation using the definition of $\psi_\omega(z)$ in terms of a power series:

$$\left|\frac{\partial^j \psi_\omega}{\partial z^j}\right|(0) = |\omega_j| \sqrt{j!}$$

Therefore: $|\omega_j| \leq c M'_{r,\omega} \sqrt{j!} r^{-|j|}$,
and using Sterling's formula ($j! \approx \sqrt{2\pi}\sqrt{j} j^j e^{-j}$), we get that:

$$|\omega_j| \leq (2\pi)^{\frac{n}{2}} (\textstyle\prod_k j_k^{\frac{1}{4}}) e^{(\frac{n}{2} - \delta) r^2 + \sum \frac{j_k}{2} \log(j_k) - (|j|)\log r - \frac{|j|}{2}}, \ \forall k, \ j_k \neq 0.$$

The $(2\pi)^{\frac{n}{2}} j^{\frac{1}{4}}$ term will not matter in the end so we will focus instead on the exponent.

$$\begin{aligned} A &= \left(\frac{n}{2} - \delta\right) r^2 - \frac{|j|}{2} + \sum_k \left(\frac{j_k}{2}\log(j_k)\right) - (|j|)\log(r) \\ &= \sum_{k=1}^{k=n} \left(\frac{j_k}{2}\right) \left(\left(1 - \frac{2\delta}{n}\right)\frac{r^2}{j_k} - 1 + \log(j_k) - 2\log(r)\right) \end{aligned}$$

Let $j_k = \gamma_k r^2$

$$\begin{aligned} A &= \sum_{k=1}^{k=n} \left(\frac{\gamma_k r^2}{2}\right)\left(\left(1-\frac{2\delta}{n}\right)\frac{1}{\gamma_k} - 1 + \log(\gamma_k)\right) \\ &= -\delta r^2 + n f(\gamma_k)\frac{r^2}{2}, \text{ where } f(\gamma_k) = 1-\gamma_k+\gamma_k\log(\gamma_k) \end{aligned}$$

$f(\gamma_k) = (1-\gamma_k)^2 - (1-\gamma_k)^3 + o((1-\gamma)^4)$ near 1.

Hence $\exists\Delta$ such that $\forall\delta \le \Delta$ if $\gamma_k \in \left[1-\sqrt{\frac{\delta}{n}}, 1+\sqrt{\frac{\delta}{n}}\right]$ then $A \le \frac{-\delta r^2}{2}$

Therefore for j as above $|\omega_j| \le (2\pi)^{\frac{n}{2}}(\prod_k j_k^{\frac{1}{4}})e^{-\frac{\delta r^2}{2}} \le cr^{\frac{n}{2}}e^{-\frac{\delta r^2}{2}}$. This holds true for all $\omega_j,\ j$ in terms of r. Specializing our work for large r, we have that $\forall\varepsilon > 0,\ \exists R$, such that $\forall r > R, |\omega_j| \le e^{-\frac{1}{2}(\delta-\varepsilon)r^2}$. Note that the factor of ε is used to compensate for the $\sqrt{2\pi}j_k^{\frac{1}{4}}$ terms. The probability of which may be estimated using Lemma 5.2 as:

$$\nu(\{\omega : |\omega_j| \le e^{-\frac{1}{2}(\delta-\varepsilon)r^2}\}) \le e^{-(\delta-\varepsilon)r^2}.$$

Hence $E_{\delta,r}$ is a small family of events as:
$\nu(\{\omega : \log M'_{r,\omega} \le (\frac{1}{2}-\delta)r^2\})$

$\le \nu(\{\omega : |\omega_j| \le e^{-\frac{1}{2}(\delta-\varepsilon)r^2},\ \textit{and}\ j_k\ \in [(1-\sqrt{\frac{\delta}{n}})r^2,\ (1+\sqrt{\frac{\delta}{n}})r^2]\})$

$\le (e^{-(\delta-\varepsilon)r^2})^{(2\sqrt{\frac{\delta}{n}}r^2)^n} = e^{-2^n(1+o(\delta))\delta^{\frac{n+2}{2}}r^{2n+2}} = e^{-c_{1,\delta}r^{2n+2}}$, using the independence of ω_j.

$M_{r,\omega} \ge M'_{\frac{1}{\sqrt{n}}r,\omega} \ge \dfrac{1}{2}r^2 - \delta r^2$, except for small events thus proving the lemma.

$\square$

Results of this type can deceive one into thinking of random holomorphic functions as $e^{\frac{1}{2}z^2}$. This absolutely is not the case, as they are weakly invariant with respect to the isometries of the reduced Heisenberg group. In particular, an analog of the previous theorem holds at any point (whereas this will be false for $e^{\frac{1}{2}z^2}$).

Corollary 5.5. *For all $\delta > 0$ and $z_0 \in \overline{B(0,r)}\backslash B(0,\frac{1}{2}r)$, there exists $\zeta \in B(z_0,\delta r)$ s.t.*

$$\log|\psi_\omega(\zeta)| > \left(\frac{1}{2} - 3\delta\right)|z_0|^2$$

except for on a small family of events.

Proof. By Lemma 5.4:

$$\nu(\{\omega : \max_{z\in\partial B(0,r)} \log|\psi_\omega(z)| - \frac{1}{2}|z|^2 \leq -\delta r^2\}) \leq e^{-cr^{2n+2}}$$

By Lemma 4.3, we have that for $z_0 \in B(0,r)\backslash B(0,\frac{1}{2}r),\ z \in B(z_0,\delta r)$:

$$\nu(\{\omega : \max_{z\in\partial B(0,\delta r)} \log|\psi_\omega(z-z_0)| - \frac{1}{2}|z-z_0|^2 \leq -\delta(\delta r)^2\}) \leq e^{-cr^{2n+2}}$$

Hence, $\exists\ z \in B(z_0,\delta r)$ s.t. $\log|\psi_\omega(z-z_0)| - \frac{1}{2}|z-z_0|^2 \geq -\delta(\delta r)^2$, except for a small family of events.

By hypothesis, $|z_0| \in [\frac{1}{2}r, r)$, hence $|z-z_0| \leq \delta r \leq \frac{1}{4}r = \frac{r}{2}\frac{1}{2} \leq \frac{1}{2}|z_0|$

Hence, $|z_0-z|^2 \geq |z_0|^2 - \delta r^2 \geq |z_0|^2(1-2\delta)$

Without loss of generality assume that $\delta < \frac{1}{4}$.

$$\begin{aligned}\log|\psi_\omega(z-z_0)| &\geq \tfrac{1}{2}|z-z_0|^2 - \delta^3 r^2 \geq |z_0|^2\tfrac{1}{2}(1-2\delta)^2 - 4\delta^3|z_0|^2\\ &\geq \tfrac{1}{2}|z_0|^2 - 2\delta|z_0|^2 - \tfrac{1}{4}\delta|z_0|^2\\ &\geq \tfrac{1}{2}|z_0|^2 - 3\delta|z_0|^2\end{aligned}$$

And, setting $\zeta = z - z_0$ this is what we set out to prove. □

Using that $\log \max\limits_{B(0,r)}|\psi_\omega|$ is an increasing function in terms of r, we have the following corollary:

Corollary 5.6. *For all $\delta > 0$*

a) $Prob\left(\left\{\omega : \lim\limits_{r\to\infty} \dfrac{(\log\max_{z\in B(0,r)}|\psi_\omega(z)|) - \frac{1}{2}r^2}{r^2} \notin [-\delta,\delta]\right\}\right) = 0$

b) $Prob\left(\left\{\omega : \lim\limits_{r\to\infty} \dfrac{(\log\max_{z\in B(0,r)}|\psi_\omega(z)|) - \frac{1}{2}r^2}{r^2} \neq 0\right\}\right) = 0$

This corollary as well as corollary 6.6 have already been proven by more direct methods, [17].

Proof. Part b follows immediately from part a, which we now prove:

Let $E_{\delta,R} = \{\omega : \frac{\log\max_{B(0,R)}\psi_\omega(z) - \frac{1}{2}R^2}{R^2} \notin [-\delta,\delta]\}$

Let $R_m = r + \delta(m+1)r,\ r > 0$

Let $s_m \in [R_{m-1}, R_m]$.

Claim: $\forall m > M_\delta,\ \forall s_m,\ E_{\delta,s_m} \subset E_{\frac{1}{3}\delta,R_m} \bigcup E_{\frac{1}{3}\delta,R_{m-1}}$

Let $M_\delta = \max\{M_{1,\delta},\ M_{2,\delta}\}$, which may be specifically determined.

Case i: for $\omega \in E_{\delta,s_m}$, $\log \max\limits_{B(0,s_m)} \psi_\omega \geq \dfrac{1}{2}s_m^2 + \delta s_m^2$

$$\begin{aligned}\log \max_{B(0,R_m)} |\psi_\omega| &\geq \tfrac{1}{2}s_m^2 + \delta s_m^2,\\ &\geq \tfrac{1}{2}(1+m\delta)^2 r^2 + \delta(1+m\delta)^2 r^2\\ &> \tfrac{1}{2}R_m^2 + \tfrac{1}{3}\delta R_m^2,\ \forall m > M_{1,\delta}\end{aligned}$$

Therefore, $\omega \in E_{\frac{\delta}{3},R_m}$

Case ii: for $\omega \in E_{\delta,s_m}$, $\log \max\limits_{B(0,s_m)} \psi_\omega \leq \dfrac{1}{2}s_m^2 - \delta s_m^2$

$$\begin{aligned}\log \max_{B(0,R_{m-1})} |\psi_\omega| &\leq \tfrac{1}{2}s_m^2 - \delta s_m^2\\ &\leq \tfrac{1}{2}(1+(m-1)\delta)^2 r^2 - \delta(1+m\delta)^2 r^2\\ &\leq \tfrac{1}{2}R_{m-1}^2 - \tfrac{1}{3}\delta R_{m-1}^2,\ \forall m > M_{2,\delta}\end{aligned}$$

Therefore, $\omega \in E_{\frac{\delta}{3},R_{m-1}}$,

Hence, $\forall m > M_\delta$ and $\forall s \in [R_{m-1}, R_m]$, $E_{\delta,s} \subset E_{\frac{1}{3}\delta,R_{m-1}} \cup E_{\frac{1}{3}\delta,R_m}$

Hence, $Prob(\bigcup_{s\in[R_{m-1},R_m]} E_{\delta,s}) \leq 2e^{-c_\delta r^{2n+2} m^{2n+2}}$, and

$\displaystyle\sum_{m\in\mathbb{N}} Prob(\bigcup_{s\in[R_{m-1},R_m]} E_{\delta,s}) = \sum_{m\in\mathbb{N}} e^{-c_\delta m^{2n+2}} < \infty$, and the result follows.

$\square$

6. **The Second main lemma.**

Essentially to prove the main theorem that we are working towards we need only one more interesting lemma, Lemma 6.5, in which we will give an estimate for $\int \log|\psi_\omega|$. This will be proved first by obtaining a crude estimate for $\int |\log|\psi_\omega||$, except for a small family of events, and then by proving facts about the Poisson Kernel, which will allow me to approximate using Riemann integration the first integral with values of $\log|\psi_\omega(z)|$ at a number of fairly evenly spaced points.

In order to establish notation I state the following standard result:

Proposition 6.1. *For $\zeta \in B(0,r)$, h a harmonic function*

$$h(\zeta) = \int_{\partial B(0,r)} P_r(\zeta, z) h(z) d\sigma_r(z)$$

where $d\sigma_r$ is the Haar measure of the sphere $S_r = \partial B(0,r)$ and P_r is the Poisson kernel for $B(0,r)$.

A proof of this can be found in many standard text books, [8]. It is convenient to normalize σ_r so that $\sigma_r(S_r) = 1$. For this normalization, the Poisson Kernel is:

$$P_r(\zeta, z) = r^{2n-2}\frac{(r^2 - |\zeta|^2)}{|\zeta - z|^{2n}}$$

Lemma 6.2. *For all $r > R_n$, $\int_{\partial B(0,r)} |\log(|\psi_\omega|)| d\sigma_r(z) \leq (3^{2n}+1)r^2$ except for a small family of events.*

Proof. By Lemma 5.4, with the exception of a small family of events, there exists $\zeta_0 \in \partial B(0, \frac{1}{2}r)$ such that $\log(|\psi_\omega(\zeta_0)|) > 0$.

Combining this with Proposition 6.1,

$$\int_{\partial B(0,r)} P_r(\zeta_0, z)\log(|\psi_\omega(z)|)d\sigma_r(z) \geq \log(|\psi(\zeta_0)|) \geq 0.$$

Hence,

$$\int_{\partial(B(0,r))} P_r(\zeta_0, z)\log^-(|\psi_\omega(z)|) \leq \int_{\partial(B(0,r))} P_r(\zeta_0, z)\log^+(|\psi_\omega(z)|)$$

Since $\zeta \in \partial B(0, \frac{1}{2}r)$ and $z \in \partial B(0, r)$, we have: $\frac{1}{2}r \leq |z - \zeta| \leq \frac{3}{2}r$. Hence by using the formula for the Poisson Kernel,

$$\frac{1}{3}\left(\frac{2}{3}\right)^{2n-2} \leq P_r(\zeta, z) \leq (2)^{2n-2}3$$

Therefore, $\int_{\partial B(0,r)} \log^+(|\psi_\omega(z)|)d\sigma_r(z) \leq \log M_r \leq (\frac{1}{2}+\delta)r^2 \leq r^2$, except for a small family of events, by Lemma 5.4.

$$\begin{aligned}
\int_{\partial(B(0,r))} P(\zeta_0, z)\log^+(|\psi_\omega(z)|) &\leq \sigma_r(S_r)\log(M_r)3(2)^{2n-2} \\
&\leq 3(2)^{2n-2}r^2 \\
\int_{\partial(B(0,r))} \log^-(|\psi_\omega(z)|)d\sigma_r(z) &\leq \frac{1}{\min_z P(\zeta_0,z)}\int_{\partial(B(0,r))} P(\zeta_0, z)\log^+(|\psi_\omega(z)|) \\
&\leq 3\left(\tfrac{3}{2}\right)^{2n-2}\int_{\partial(B(0,r))} P(\zeta_0, z)\log^+(|\psi_\omega(z)|) \\
&\leq 9\left(\tfrac{3}{2}\right)^{2n-2}(2)^{2n-2}r^2 \\
&\leq 3^{2n}r^2
\end{aligned}$$

And, the result follows immediately.

□

As we are already able to approximate $\log|\psi_\omega(z)|$ at any finite number of points in order to use Reimann integration to prove Lemma 6.5 we will need to be able to choose "evenly" spaced points on the sphere, as chosen according to the next proposition:

Lemma 6.3. *(A partition of a Sphere)*
If $(2n)m^{2n-1} = N$
then $S_r^{2n} \subseteq \mathbb{R}^{2n}$ *can be "divided" into measurable sets* $\{I_1^r, I_2^r, \ldots, I_N^r\}$ *such that:*

1) $\bigcup_j I_j^r = S_r$
2) $\forall j \neq k,\ I_j^r \bigcap I_k^r = \emptyset,$
3) $diam(I_j^r) \leq \frac{\sqrt{2n-1}}{m} r = \frac{c_n}{N^{\frac{1}{2n-1}}} r$

Proof. Surround S_r with 2n pieces of planes: $P_{+,1}, P_{+,2}, \ldots, P_{+,n}, P_{-,1}, \ldots P_{-,n}$, where

$$P_{+,j} = \{x \in \mathbb{R}^{2n+1} :\ ||x||_{L^\infty} = r,\ x_j = r\}$$

$$P_{-,j} = \{x \in \mathbb{R}^{2n+1} :\ ||x||_{L^\infty} = r,\ x_j = -r\}$$

Subdivide each piece into m^{2n-1} even $2n-1$ cubes, in the usual way, and denote these sets $R_1, \ldots,\ R_N$.

Let $I_j^r = \{x \in S_r :\ \lambda x \in R_n,\ \lambda > 0\}$

By design, $\lambda \geq 1$ and $x,\ y \in I_j \Rightarrow\ d(x,y) < d(\lambda_1 x, \lambda_2 y) \leq \frac{2}{m} r = diam(R_j)$. These sets can be redesigned to get that $I_j^r \bigcap I_k^r = \emptyset$, $j \neq k$ by carefully defining R_i so that $R_j \bigcap R_k = \emptyset$. ☐

The following elementary result is less well known then others and will be very useful in proving Lemma 6.5. Note this integration is with respect to w, which is not the same variable of integration that is used in Proposition 6.1. This is done because the goal of this section, Lemma 6.5, is to estimate a surface integral, which corresponds to integration with respect to the first variable.

Lemma 6.4. *For* $\kappa < 1$

$$\int_{w \in S_{\kappa r}^n} P_r(w,z) d\sigma_{\kappa r}(w) = 1$$

Proof. $P_r(w,z) = r^{2n-2} \frac{r^2 - |w|^2}{|z-w|^{2n}},\ z \in \partial B(0,r)$

If $w \in S_{\kappa r}^{2n} \subseteq \mathbb{R}^{2n}$, then the poisson Kernel can be rewritten as a function of $|z-w|$, and as such $\forall \Upsilon \in U_n(\mathbb{R}^n),\ P_r(\Upsilon w, \Upsilon z) = P_r(w,z)$

Let $f(z) = \int_{w \in S_{\kappa r}^n} P_r(w,z) d\sigma_{\kappa r}(w)$

$$\begin{aligned} f(z) &= \int_{w\in S^n_{\kappa r}} P_r(w,z)d\sigma_{\kappa r}(w)\\ &= \int_{w\in S^n_{\kappa r}} P_r(\Upsilon w,\Upsilon z)d\sigma_{\kappa r}(w), \text{ by the above work.}\\ &= \int_{w\in S^n_{\kappa r}} P_r(\Upsilon w,\Upsilon z)d\sigma_{\kappa r}(\Upsilon w), \text{ as } d\sigma_{\kappa r} \text{ is invariant under rotations.}\\ &= \int_{w\in S^n_{\kappa r}} P_r(w,\Upsilon z)d\sigma_{\kappa r}(w), \text{ by a change of coordinates.}\\ &= f(\Upsilon z) \end{aligned}$$

Hence $f(z) = c,\ \forall z \in S^n_r$
By switching the order of integration we compute that:

$$1 = \int_{w\in S^n_{\kappa r}} \int_{z\in S^n_r} P_r(w,z)d\sigma_r(z)d\sigma_{\kappa r}(w) = c$$

□

Now we are able to prove our final lemma.

Lemma 6.5. *For all* $\Delta > 0$,

$$\left\{\omega : \tfrac{1}{r^2}\int_{z\in\partial B(0,r)} \log|\psi_\omega| d\sigma_r(z) \leq \frac{1}{2} - \Delta\right\} \text{ is a small family of events.}$$

Proof. It suffices to prove the result for small Δ. Let $\Delta > 0$. Let
$a_n = \frac{1}{2(2n+2)(2n-1)}$. Set $\delta = \left(\frac{1}{\lambda}\Delta\right)^{(\frac{1}{a_n})} < \frac{1}{6},\ \lambda > 0$ to be determined later.
Choose $m \in \mathbb{N}$ such that writing $N = (2n)m^{2n-1},\ \frac{1}{N} \leq \delta$. Let $\kappa = 1 - \delta^{a_n}$.

Choose $I^{\kappa r}_j$ measurable subsets of $S_{\kappa r}$ as in Proposition 6.3.
In particular:

1) $S_{\kappa r} = \cup I^{\kappa r}_j$, a disjoint union.
2) $\sum \sigma_r(I^{\kappa r}_j) = 1$
3) $diam(I^{\kappa r}_j) \leq \frac{c_n}{N^{\frac{1}{2n-1}}}\kappa r \leq c\delta^{\frac{1}{2n-1}} r$

Let $\sigma_j = \sigma_{\kappa r}(I^{\kappa r}_j)$, which does not depend on r.

For all j fix a point $x_j \in I^{\kappa r}_j$

By Lemma 5.5, $\exists \zeta_j \in B(x_j, \delta r)$ such that

$$\log(|\psi_\omega(\zeta_j)|) > \left(\frac{1}{2} - 3\delta\right)|x_j|^2 = \left(\frac{1}{2} - 3\delta\right)\kappa^2 r^2$$

Except, of course, on N different small families of events (the union of which remains a small family of events).

$$
\begin{aligned}
(\frac{1}{2}-3\delta)\quad (1-\delta^{a_n})^2r^2 &\leq \sum_{j=1}^{N}\sigma_j\log(|\psi_\omega(\zeta_j)|)\\
&\leq \int_{\partial B(0,r)}\left(\sum_j \sigma_j P_r(\zeta_j,z)\log(|\psi_\omega(z)|)d\sigma_r(z)\right)\\
&= \int_{\partial(B(0,r))}\left(\sum_j \sigma_j(P_r(\zeta_j,z)-1)\right)\log(|\psi_\omega(z)|)d\sigma_r(z)\\
&\quad+\int_{\partial(B(0,r))}\log(|\psi_\omega(z)|)d\sigma_r(z)
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\int_{\partial B(0,r)} \quad & \log(|\psi_\omega|)d\sigma_r\\
& \geq (\tfrac{1}{2}-3\delta)(1-\delta^{a_n})^2r^2-\int|\log|\psi_\omega||d\sigma_r\cdot\max_z|\textstyle\sum_j\sigma_j(P_r(\zeta_j,z)-1)|\\
& \geq (\tfrac{1}{2}-3\delta)(1-\delta^{a_n})r^2-(3^{2n}+1)r^2\cdot C_n\delta^{\frac{1}{2(2n-1)}}\geq\tfrac{1}{2}r^2-\lambda\delta^{a_n}r^2
\end{aligned}
$$

by Lemmas 6.2 and the following claim. After proving this claim, the result will follow.

Claim: $\max_{z\in\partial(B(0,r))}\left|\sum_j \sigma_j(P_r(\zeta_j,z)-1)\right|\leq C_n\delta^{\frac{1}{2(2n-1)}}$

Proof of claim: $\forall z\in\partial B(0,r)$, $\int_{\zeta\in\partial B(0,\kappa r)}P_r(\zeta,z)d\sigma_{\kappa r}(\zeta)=1$, by Lemma 6.4.

Hence, $1=\sum_{j=1}^{j=N}\sigma_jP_r(\zeta_j,z)+\sum_{j=1}^{j=N}\int_{\zeta\in I_j^{\kappa r}}(P_r(\zeta,z)-P_r(\zeta_j,z))d\sigma_{\kappa r}(\zeta)$

$$
\begin{aligned}
\text{And, } |\textstyle\sum_{j=1}^{j=N}\quad \sigma_j(P_r(\zeta_j,z)-1)| &= |\textstyle\sum_{j=1}^{j=N}\int_{\zeta\in I_j^{\kappa r}}(P_r(\zeta,z)-P_r(\zeta_j,z))d\sigma_{\kappa r}(\zeta)|\\
&\leq \max_{j,\ \zeta\in I_j^{\kappa r}}|\zeta-\zeta_j|\cdot\max_{w\in B(0,(\kappa+\delta)r)\setminus B(0,(\kappa-\delta)r)}\left|\frac{\partial P_r(w,z)}{\partial w}\right|
\end{aligned}
$$

$$\frac{\partial P_r(w,z)}{\partial w}=-r^{2n-2}\frac{\overline{w}|z-w|^2+(r^2-|w|^2)n(\overline{z}-\overline{w})}{|z-w|^{2n+2}}$$

As $|z|=r$, and $|w|=(1-\varepsilon)r\in[(\kappa-\delta)r,(\kappa+\delta)r]$

$\left|\frac{\partial P_r(w,z)}{\partial w}\right|\leq\frac{2+4\varepsilon n}{r\varepsilon^{2n+2}}\leq\frac{c_n}{r\varepsilon^{2n+2}}=\frac{c_n}{r}\delta^{-\frac{1}{2(2n-1)}}$

And, $\max_\zeta|\zeta-\zeta_j|\leq diam(I_j)+\delta r\leq c\delta^{\frac{1}{2n-1}}r+\delta r\leq c'r\delta^{\frac{1}{2n-1}}$

Therefore: $|\sum_{j=1}^{j=N}\sigma_j(P_r(\zeta_j,z)-1)|\leq C\delta^{\frac{1}{2n-1}}\cdot\delta^{-\frac{1}{2(2n-1)}}=C\delta^{\frac{1}{2(2n-1)}}$

Proving the claim and the lemma.

$\square$

This lemma gives an alternate proof for the growth rate of the characteristic function. Let $T(f,r) = \int_{S_r} \log^+ |f(z)| d\sigma_r(z)$, the Nevanlina characteristic function. As $(\int_{S_r} \log|\psi_\omega| d\sigma_r)$ is increasing the proof of Corollary 5.6 can be used in conjunction with Lemma 6.5 to prove that $\psi_\omega(z)$ is a.s. finite order 2.

Corollary 6.6. *For all* $\delta \in (0, \frac{1}{3}]$

$$a)\ \ Prob\left(\left\{\omega : \lim_{r\to\infty} \frac{(\int_{S_r} \log|\psi_\omega| d\sigma_r) - \frac{1}{2}r^2}{r^2} \notin [-\delta, \delta]\right\}\right) = 0$$

$$b)\ \ Prob\left(\left\{\omega : \lim_{r\to\infty} \frac{(\int_{S_r} \log|\psi_\omega| d\sigma_r) - \frac{1}{2}r^2}{r^2} \neq 0\right\}\right) = 0$$

$$c)\ \ Prob\left(\left\{\omega : \lim_{r\to\infty} \frac{T(\psi_\omega, r) - \frac{1}{2}r^2}{r^2} \neq 0\right\}\right) = 0$$

7. **Proof of Main results.**

We will now be able to put the pieces together to estimate the number of zeroes in a large ball for a random holomorphic function $\psi_\omega(z)$. Further, This will help us to compute the hole probability.

Definition 7.1. *For* $f \in \mathcal{O}(B(0,r)),\ B(0,r) \subset \mathbb{C}^n$, *the unintegrated counting function,*
$n_f(r) := \int_{B(0,t)\bigcap Z_f} (\frac{i}{2\pi}\partial\overline{\partial}\log|z|^2)^{n-1} = \int_{B(0,t)} (\frac{i}{2\pi}\partial\overline{\partial}\log|z|^2)^{n-1} \wedge \frac{i}{2\pi}\partial\overline{\partial}\log|f|$

The equivalence of these two definitions follows by the Poincare-Lelong formula. The above form $((\frac{i}{2\pi}\partial\overline{\partial}\log|z|^2)^{n-1})$ gives a projective volume, with which it is more convenient to measure the zero set of a random function. The Euclidean volume may be recovered as $\int_{B(0,t)\bigcap Z_f} (\frac{i}{2\pi}\partial\overline{\partial}\log|z|^2)^{n-1} = \int_{B(0,t)\bigcap Z_f} (\frac{i}{2\pi t^2}\partial\overline{\partial}|z|^2)^{n-1}$.

Lemma 7.2. *If* $u \in L^1(\overline{B}_r)$, *and* $\partial\overline{\partial}u$ *is a measure, then*

$$\int_{t=r\neq 0}^{t=R} \frac{dt}{t} \int_{B_t} \frac{i}{2\pi}\partial\overline{\partial}u \wedge (\frac{i}{2\pi}\partial\overline{\partial}\log|z|^2)^{m-1} = \frac{1}{2}\int_{S_R} u d\sigma_R - \frac{1}{2}\int_{S_r} u d\sigma_r$$

A proof of this result is available in the literature, [16].

When applying this to random functions, my previous estimates of the surface integral will turn out to be extremely valuable.

Theorem 1.1 For all $\delta > 0$,

$$F_r := \left\{\omega : \left| n_{\psi_\omega}(r) - \frac{1}{2}r^2 \right| \geq \delta r^2 \right\} \ \textit{is a small family of events.}$$

Proof. It suffices to prove the result for small δ.

We will start by estimating that:

$$\nu\left(\left\{\omega : \frac{n_{\psi_\omega}(r)}{r^2} \geq \frac{1}{2} + \delta\right\}\right) \leq e^{-c_\delta r^{2n+2}}$$

$n_{\psi_\omega}(r)\log(\kappa) \leq \int_{t=r}^{t=\kappa r} n_{\psi_\omega}(t)\frac{dt}{t} \leq n_{\psi_\omega}(\kappa r)\log(\kappa)$, as n(r) is increasing.

let $\kappa = 1 + \sqrt{\delta}$. Except for a small family of events, we have:

$$\begin{aligned}
n_{\psi_\omega}(r)\log(\kappa) &\leq \int_{t=r}^{t=\kappa r} n_{\psi_\omega}(t)\frac{dt}{t} \\
&= \int_{t=r}^{t=\kappa r} \int_{B(0,t)} \frac{i}{2\pi}\partial\overline{\partial}\log|\psi_\omega(z)| \wedge \left(\frac{i}{2\pi}\partial\overline{\partial}\log|z|^2\right)^{n-1} \frac{dt}{t} \\
&= \frac{1}{2}\int_{S_{\kappa r}} \log|\psi_\omega(z)|d\sigma - \frac{1}{2}\int_{S_r} \log|\psi_\omega(z)|d\sigma, \text{ by Lemma 7.2.} \\
&\leq \frac{1}{2}\left(\left(\frac{1}{2}+\delta\right)\kappa^2 r^2 - \int_{S_r} \log|\psi_\omega(z)|d\sigma\right), \text{ by Lemma 5.4.} \\
&\leq \frac{1}{2}\left(\left(\frac{1}{2}+\delta\right)r^2\kappa^2 - \left(\frac{1}{2}-\delta\right)r^2\right), \text{ by Lemma 6.5.} \\
2\frac{n_{\psi_\omega}(r)}{r^2} &\leq \frac{1}{\log(\kappa)}\left(\kappa^2\left(\frac{1}{2}+\delta\right) - \left(\frac{1}{2}-\delta\right)\right) \\
&= \frac{\kappa^2-1}{2\log(\kappa)} + \delta\frac{\kappa^2+1}{\log(\kappa)} \leq 1 + c\sqrt{\delta}.
\end{aligned}$$

This proves the probability estimate when the unintegrated counting function is significantly larger then expected.

In order to prove the other probability estimate:

$$\nu\left(\left\{\omega : \frac{n_{\psi_\omega}(r)}{r^2} \leq \frac{1}{2} - \delta\right\}\right) \leq e^{-c_\delta r^{2n+2}}$$

We start by using that: $\int_{t=\kappa^{-1}r}^{t=r} n_{\psi_\omega}(t)\frac{dt}{t} \leq n_{\psi_\omega}(r)\log(\kappa)$. We then use that, except for a small family of events, we have that:

$$
\begin{aligned}
n_{\psi_\omega}(r)\log(\kappa) &\geq \int_{t=\kappa^{-1}r}^{t=r} n_{\psi_\omega}(t)\frac{dt}{t} \\
&= \int_{t=\kappa^{-1}r}^{t=r}\int_{B(0,t)} \frac{i}{2\pi}\partial\overline{\partial}\log|\psi_\omega(z)| \wedge \left(\frac{i}{2\pi}\partial\overline{\partial}\log|z|^2\right)^{n-1}\frac{dt}{t} \\
&= \frac{1}{2}\int_{S_r}\log|\psi_\omega(z)|d\sigma - \frac{1}{2}\int_{S_{\kappa^{-1}r}}\log|\psi_\omega(z)|d\sigma, \text{ by Lemma 7.2.} \\
&\geq \frac{1}{2}[(\frac{1}{2}-\delta)r^2 - \int_{S_{\kappa^{-1}r}}\log|\psi_\omega(z)|d\sigma], \text{ by Lemma 6.5.} \\
&\geq \frac{1}{2}[(\frac{1}{2}-\delta)r^2 - (\frac{1}{2}+\delta)r^2\kappa^{-2}], \text{ by Lemma 5.4.} \\
2\frac{n_{\psi_\omega}(r)}{r^2} &\geq \frac{1}{\log(\kappa)}\left(\left(\frac{1}{2}-\delta\right) - (\frac{1}{2}+\delta)\kappa^{-2}\right) \\
&= \frac{1-\kappa^{-2}}{2\log(\kappa)} - \delta\frac{1+\kappa^{-2}}{\log(\kappa)} \geq 1-2\sqrt{\delta}
\end{aligned}
$$

□

Using this estimate for the typical measure of the zero set of a random function we get an upper bound for the hole probability, and putting this together with some elementary estimates we get an accurate estimate for the order of the decay of the hole probability:

Theorem 1.2 *If*

$$\psi_\omega(z_1, z_2 \ldots, z_n) = \sum_j \omega_j \frac{z_1^{j_1} z_2^{j_2} \ldots z_n^{j_n}}{\sqrt{j_1! \cdot j_n!}},$$

where ω_j are independent identically distributed complex Gaussian random variables, and

$$Hole_r = \{\omega : \forall z \in B(0,r),\ \psi_\omega(z) \neq 0\},$$

then there exists $c_1, c_2 > 0$ such that for all $r > R_n$

$$e^{-c_2 r^{2n+2}} \leq Prob(Hole_r) \leq e^{-c_1 r^{2n+2}}$$

Proof. The upper estimate follows by the previous theorem, as if there is a hole then $n_{\psi_\omega}(r) = 0$, and this can only occur on a small family of events.

Therefore it suffices to show that the hole probability is bigger than a small set.

Let Ω_r be the event where:

i) $|\omega_0| \geq E_n + 1$,

ii) $|\omega_j| \leq e^{-(1+\frac{n}{2})r^2},\ \forall j : 1 \leq |j| \leq \lceil 24nr^2 \rceil = \lceil (n \cdot 2 \cdot 12) r^2 \rceil$

iii) $|\omega_j| \leq 2^{\frac{|j|}{2}},\ |j| > \lceil 24nr^2 \rceil \geq 24nr^2$
$\nu(\{\omega|\ |\omega_j| \leq e^{-(1+\frac{n}{2})r^2}\}) \geq \frac{1}{2}(e^{-(1+\frac{n}{2})r^2})^2 = \frac{1}{2}e^{-(2+n)r^2}$, by Lemma 5.2

$\#\{j \in \mathbb{N}^n | 1 \leq |j| \leq \lceil 24nr^2 \rceil\} = (\binom{\lceil 24nr^2 \rceil + n}{n}) \approx cr^{2n}$

Hence, $\nu(\Omega_r) \geq C(e^{-c_n r^{2n+2}})$, by independence and Lemma 5.2. Therefore Ω_r contains a small family of events, and it now suffices to show that for $\omega \in \Omega_r,\ \psi_\omega$ has a hole in $B(0,r)$.

$$f(z) \geq |\omega_0| - \sum_{|j|=1}^{|j|\leq\lceil 24nr^2 \rceil} |\omega_j| \frac{r^{|j|}}{\sqrt{j!}} - \sum_{|j|>\lceil 24nr^2 \rceil} |\omega_j| \frac{r^{|j|}}{\sqrt{j!}} = |\omega_0| - \sum^1 - \sum^2$$

$$\begin{aligned}
\sum^1 &\leq e^{-(1+\frac{n}{2})r^2} \sum_{|j|=1}^{|j|\leq\lceil 24nr^2 \rceil} \frac{r^{|j|}}{\sqrt{j!}} \\
&\leq e^{-(1+\frac{n}{2})r^2} \sqrt{(24nr^2+1)^n}\sqrt{(e^{r^n})}, \text{ by Cauchy-Schwarz inequality.} \\
&\leq C_n r^n e^{-r^2} \leq ce^{-0.9r^2} < \tfrac{1}{2} \text{ for } r > R_n
\end{aligned}$$

$$\begin{aligned}
\sum^2 &\leq \sum_{|j|>24nr^2} 2^{\frac{|j|}{2}} \left(\frac{|j|}{24n}\right)^{\frac{|j|}{2}} \frac{1}{\sqrt{j!}}, \text{ as } r < \sqrt{\tfrac{|j|}{24n}} \\
&\leq c \sum_{|j|>24nr^2} 2^{\frac{|j|}{2}} \left(\frac{|j|}{24n}\right)^{\frac{|j|}{2}} \prod_{k=1}^{k=n} \left(\frac{e}{j_k}\right)^{\frac{j_k}{2}}, \text{ by Sterling's formula} \\
&= c \sum_{|j|>24nr^2} \frac{(|j|)^{\frac{|j|}{2}}}{\left(\prod_{k=1}^{k=n} j_k^{\frac{j_k}{2}}\right) n^{\frac{|j|}{2}}} \left(\frac{e}{12}\right)^{\frac{|j|}{2}} \\
&\leq c \sum_{|j|>1} \left(\frac{1}{4}\right)^{\frac{|j|}{2}}, \text{ by Lemma 5.3.} \\
&\leq c \sum_{l>1} \left(\frac{1}{2}\right)^l l^n \leq E_n
\end{aligned}$$

Hence, $|\psi_\omega(z)| \geq E_n + 1 - \sum^1 - \sum^2 \geq \dfrac{1}{2}$ □

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